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On The Norm of Jordan Elementary Operator

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Abstract

Many researchers in operator theory have studied the norm of Jordan elementary operator. Various results have been obtained using various approaches to establish the lower bound of this norm. In this paper we attempt the same problem for finite dimensional operators.

Keywords: Norm, Jordan Elementary operator.

Introduction

Properties of Elementary Operators have been investigated in the recent past under various aspects. Their norms have been a subject of interest for research in operator theory. Deriving a formula to express the norm of Elementary Operator in terms of its coefficient operators remain a topic for research in operator theory.

Let H be a complex Hilbert space and $B(H)$ the algebra of bounded linear operators on H. The Jordan Elementary Operator $U_{A,B} : B(H) \rightarrow B(H)$ is defined by

$$U_{A,B}(X) = AXB + BXA \qquad (1)$$

Where $X \in B(H)$ with $A, B \in B(H)$ fixed. Some other elementary operators are;

(i) The basic elementary operator $M_{A;B}$ defined by $M_{A,B}(X) = AXB$.

(ii) Left multiplication operator L_A defined by $L_A(X) = AX$ and Right multiplication operator R_A defined by $R_A(X) = XA$

(iii) The multiplication operator $\Delta_{A;B}$ represented as $\Delta_{A;B} = L_A R_B$

Kingangi, Agure and Nyamwala [6] in 2014 used maximal numerical range to prove that $\|U_{A,B}\|=2\|A\|\|B\|$ Xiaolo [4] in 2008 showed that $\|U_{A,B}\| = \|A\|\|B\|$ by identifying B(H) with 2 × 2 complex matrices and H 2 dimensional.

Stacho and Zalar [3], in 1996 showed that the lower bound of $U_{A,B}$ to be estimated as $\|U_{A,B}\| \geq 2(\sqrt{2} - 1)\|A\|\|B\|$ using standard operator algebras acting on a Hilbert space H. Later in 1998, they used algebra of symmetric operators acting on a Hilbert space H and showed that $\|U_{A,B}\| \geq \|A\|\|B\|$.

On their part, Cabrera and Rodriguez [2] in 1994 Showed that $\|U_{A,B}\| \geq {}_{1/204012} \|A\|\|B\|$ for JB*- algebras.

Mathew [1] in 1990 proved that $\|U_{A,B}\| \geq 2/3\|A\|\|B\|$ for prime C*-algebra. In this paper we use finite rank operators to determine the norm of Jordan Elementary Operator.

The norm of elementary operator

In this section, we present some known results on elementary operators.

Theorem 1: Let H be a complex Hilbert space and B(H) the algebra of bounded linear operators on H. Let MA,B : B(H) → B(H) be defined by MA,B(X) = AXB for all X ∈B(H) with A, B ∈B(H) fixed and //X // = 1. If X(x) = x for all unit vectors x ∈H, then //MA,B // = //A // //B //. See [5] for the proof.

In the following theorem, King'angi, Agure and Nyamwala [6] determined the norm of elementary operator of length two.

Theorem 2: Let H be a complex Hilbert space and B(H) the algebra of all bounded linear operators on H Let Δ_2 : B(H) → B(H) be an elementary operator on B(H) defined by $\Delta_2(X) = \Sigma^2_{i=1} A_1XB_1 + A_2XB_2$ for all X ∈ B(H) with A_i, B_i ∈ B(H) fixed for i = 1, 2. If X(x) = x for all unit vectors x ∈ H, then;

$\|\Delta_2\| = \Sigma^2_{i=1} \|A_i\|\|B_i\|$.

Main Result

The following theorem is the main result in this paper.

Theorem 3: Let H be a complex Hilbert space and B(H) the algebra of all bounded linear operators on H. Let $U_{A,B}$ be the Jordan elementary operator on B(H) defined as in (1) above. Let X ∈ B(H) with ‖X‖ = 1 and the unit vector x ∈ H such that X(x) = x then: $\|U_{A,B/B(H)}\| = 2\|A\|\|B\|$.

Proof: By definition, $\|U_{A,B/B(H)}\| = \sup\{\|U_{A,B}(X)\| : X \in B(H), \|X\| = 1\}$.

Thus $\|U_{A,B/B(H)}\| \geq \|U_{A,B}(X)\|$ for all X ∈ B(H) with ‖X‖ = 1.

Taking $\epsilon > 0$ we have,

$\|U_{A,B/B(H)}\| - \epsilon < \|U_{A,B}(X)\|$ for all X ∈ B(H) with ‖X‖ = 1

$= \|AXB + BXA\|$

$\leq \|AXB\| + \|BXA\|$

$= \|A\|\|X\|\|B\| + \|B\|\|X\|\|A\|$

$= \|A\|B\| + \|B\|\|A\|$

$= 2\|A\|\|B\|$

Letting $\epsilon \to 0$, we have

$$\|U_{A,B/B(H)}\| \leq 2\|A\|\|B\|. \quad (2)$$

We now show that $\|U_{A,B/B(H)}\| \geq 2\|A\|\|B\|$.

We let f and g be functionals on H and choose unit vectors $y, z \in H$.

Let $A = f \otimes y$ and $B = g \otimes z$ be finite rank operator on H defined by

$Ax = (f \otimes y)x = f(x)y$ and $Bx = (g \otimes z)x = g(x)z$ respectively, for all $x \in H$ with $//x// = 1$.

The norm of $A \in B(H)$ is given as follows;

$\|A\| = \sup\{\| (f \otimes y)x\| : x \in H, \|x\| \leq 1\}$

$= sup\{|f(x)| : x \in H, \leq 1\}$

$= |f(x)|$.

Similarly, $\|B\| = |g(x)|$ for any unit vector $x \in H$ with $\|x\| = 1$.

Since $\|U_{A,B}(X)\| = \sup\{\| U_{A,B}(X)x\| : x \in H, \|x\| = 1\}$,

then we have $\|U_{A,B}(X)\| \geq \|U_{,B}(X)x\|$ for all $x \in H$ with $\|x\| = 1$ (3)

But $U_{A,B}(X)x = (AXB + BXA)x$

$= (AXB)x + (BXA)x$

$= (f \otimes y)X(g \otimes z)x + (g \otimes z)X(f \otimes y)x$

$= (f \otimes y)Xg(x)z + (g \otimes z)Xf(x)y$

$= g(x)(f \otimes y)X(z) + f(x)(g \otimes z)X(y)$

$= g(x)f(X(z))y + f(x)g(X(y))z$.

From (3) $\|U_{A,B/B(H)}\| = \sup\{\| U_{AB}(X)\| : X \in B(H), \|X\| = 1\}$. So

$\|U_{A,B/B(H)}\| \geq \|U_{A,B}(X)\| : X \in B(H), \|X\| = 1\}$.

Now $\|U_{A,B/B(H)}\|^2 \geq \|g(x)f(X(z))y + f(x)g(X(y))z\|^2$

$= \langle g(x)f(X(z))y + f(x)g(X(y))z, g(x)f(X(z))y + f(x)g(X(y))z\rangle$

$= \|g(x)f(X(z))y\|^2 + \langle g(x)f(X(z))y, f(x)g(X(y))z\rangle + \langle f(x)g(X(y))z, g(x)f(X(z))y\rangle + \|f(x)g(X(y))z\|^2$

$= \|g(x)f(X(z))y\|^2 + g(x)f(X(z))f(x)g(X(y))\langle y, z\rangle + f(x)g(X(y))g(x)f(X(z))\langle z, y\rangle + \|f(x)g(X(y))z\|^2$

$= |g(x)|^2|f(X(z))|^2\|y\|^2 + g(x)f(X(z))f(x)g(X(y)) + f(x)g(X(y))g(x)f(X(z)) + f(x)|^2|g(X(y))|^2\|y^2$

$= \{|g(x)||f(X(z))|\}^2 + 2\{g(x)f(X(z))f(x)g(X(y))\} + \{|f(x)||g(X(y))|\}^2$

Since g(x), f(x), f(X(z)) and g(X(y)) are all positive numbers, by taking;

$g(x) = |g(x)| = \|B\|$ and $g(X(y)) = |g(X(y))| = \|B\|$,

$f(x) = |f(x)| = \|A\|$ and $f(X(z)) = |f(X(z))| = \|A\|$.

We therefore have, $\|U_{A,B/B(H)}\|^2 \geq \{ B\|\|A \}^2 + 2\|B\|\|A\|\|A\|\|B\| + \{A\|\|B \}^2$

$= \{(\|A\|\|B\|) + (\|A\|\|B\|)\}^2$

Taking square root on both sides and adding we obtain;

$\|U_{A,B/B(H)}\| \geq 2\|A\|\|B\|$. (4)

So by (2) and (4) we have that

$\|U_{A,B/B(H)} = 2\|A\|\|B\|$.

Thus $\|U_{A,B}\| = 2\|M_{A;B}\|$

This completes the proof.

Conclusion

The Norm of Jordan Elementary operator is equal to twice the norm of Basic elementary operator.

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